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# Nonlinear Control of Flexible, Articulated Spacecraft: Application to Space Station/Mobile Manipulator

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**This paper extends the authors' prior work on the attitude control of flexible space structures via partial feedback linearization methods to articulated systems. Recursive relations introduced by Jain and Rodriguez are central to the efficient formulation of models via Poincaré's form of Lagrange's equations. Such models provide for easy construction of feedback linearizing control laws. Adaptation is shown to be an effective way of reducing sensitivity to uncertain parameters. An application to a flexible platform with mobile remote manipulator system is highlighted.**

## I. Introduction

OUR goal is to demonstrate the application of recent innovations for modeling and control of articulated systems to a spacecraft configuration representative of Space Station Freedom with a mobile remote manipulator system (SSF/MRMS). The problem considered is the attitude regulation of the space station while the MRMS undergoes arbitrary prescribed maneuvers. The issue of attitude control for such a configuration has received attention in the literature, most notably in the papers of Mah et al.[1] and Wie et al.,[2] in which linear controllers are applied and various stability problems are noted. Such systems are inherently nonlinear, and this raises questions both with respect to modeling, especially when flexibility is present, and with respect to control system design.

The methods considered here address the essential nonlinearity of these systems directly. A unified approach to modeling and nonlinear control system design is employed. In recent work, including Ref. 3, we considered the design of feedback linearizing controls for multibody, flexible, but nonarticulated, spacecraft in connection with the NASA/IEEE Spacecraft Controls Laboratory Experiment (SCOLE) problem. Here we extend those methods to articulated systems. In Secs. II and III we develop the formulation of Poincaré's equations for articulated systems with rigid and flexible bodies using the recursive constructions of Rodriguez,[4] Jain,[5] and Jain and Rodriguez.[6] In Sec. IV we derive the feedback linearizing attitude control laws for this general class of systems. Nonadaptive and adaptive controls are described. We apply these results to the SSF/MRMS configuration in Sec. V.

The space station attitude control issues addressed here are related to the attitude control problems defined by Mah et al.[1] and Wie et al.[2] except that we focus on the short time-scale problem (time scale of minutes) associated with MRMS motion, whereas in the aforementioned works MRMS induced disturbances are considered, however, primarily in terms of their effect on long-term behavior (time scale of orbits). In the latter case the important questions have to do with the ability of the attitude regulator to reject long-term periodic disturbances due to environmental torques including gravity gradient torque and cyclic aerodynamic torques. Both Refs. 1 and 2 outline the potential benefits of linear quadratic Gaussian design for this controller. They also show the sensitivity of attitude control performance to MRMS motion. In fact, Wie et al.[2] show that large MRMS motion can destabilize the attitude control system because of changes in the system inertia and suggest the need for gain scheduling of linear controllers. Since we focus on the short time scale, we do not include environment (orbital frequency) disturbance torques in our analysis.

We show that the stabilization issues are far more subtle and critical than suggested in either Refs. 1 or 2. The nonlinear inertial cross couplings, especially when platform flexibility is considered, severely limit the achievable performance with linear regulators. Because the associated dynamics are nonlinear in an essential way, we consider nonlinear control design using partial feedback linearization. This method effectively cancels certain nonlinearities and, hence, there arise important robustness issues. As a result adaptation is considered to be an important adjunct to this class of controllers. The results in this paper extends those of Ref. 3 to a larger class of systems modeled in terms of Poincaré's equations. We give an explicit characterization of the zero dynamics for this class of problems.

## II. Lagrange's Equations and Quasivelocities

Our approach to multiflex-body modeling is based on the Lagrangian framework for distributed system dynamics. The Lagrangian dynamics are conveniently formulated using quasivelocities[7-9] that result in a system of equations often called Poincaré's equations. The method has been further developed using finite element analysis for reduction to finite dimensions and the recursive constructions introduced in Refs. 4–6 for serial chains of articulating bodies. The resultant equations are convenient for analysis, computation, and control system design.

### A. Hamilton's Principle and the Euler-Lagrange Equations

The formalism of Lagrangian dynamics begins with the identification of the configuration space, i.e., the generalized coordinates, associated with the dynamical system of interest. Once the configuration manifold $M$ is specified, there follows the natural definition of velocity at a point $q \in M$ as an element

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$\dot{q}$ in the tangent space to $M$ at $q$, often denoted $T_qM$. The state space is defined as the union of tangent spaces at all points $q \in M$, the so-called tangent bundle $TM$. The evolution of the system in the state space is characterized by the definition of a Lagrangian $\mathcal{L}(q, \dot{q})$: $TM \rightarrow \mathbb{R}$ and use of Hamilton's principle of least action. The motion of a dynamical system between times $t_1$ and $t_2$ is a "natural" motion if and only if

$$\int_{t_1}^{t_2} (\delta \mathcal{L} + Q^t \delta q)\, dt = 0 \tag{1}$$

The variational statement (1) leads to the d'Alembert-Lagrange equation,[7,10,11]

$$\left\{ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} - Q^t \right\} \delta q = 0 \tag{2}$$

If the coordinate variations $\delta q$ are independent, then Eq. (2) yields the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q \tag{3}$$

In the usual case we have $\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{V}(q)$, where $\mathcal{T}$ and $\mathcal{V}$ are the kinetic energy and potential energy functions, respectively.

In the event that the coordinates variations $\delta q$ are constrained, then the d'Alembert-Lagrange equation (2) can be used to derive alternatives to the Euler-Lagrange equations (this is usually accomplished via Maggi's equations,[11] which provide one route to Kane's equations[12]). This is the basis for the analytical mechanics of nonholonomic systems, for example, see Ref. 11.

### B. Quasivelocities and Alternate Equations of Motion

It is well known that in some cases it is easier to formulate the equations of motion in terms of velocity variables that cannot be expressed as the time derivatives of any corresponding configuration coordinates. Such velocities are called quasivelocities. Quasivelocities are meaningful physical quantities—the angular velocity of a rigid body is a prime example. The notion of quasivelocities leads to a generalization of Lagrange's equations that is applicable to systems with nonholonomic as well as holonomic constraints. Such generalizations were produced at the turn of the century (see, for example, Refs. 7 and 11). A currently popular formulation is due to Kane and Levinson.[13]

Let $M$ be the $m$-dimensional configuration manifold for a Lagrangian system and suppose $v_1, \ldots, v_m$ constitute a system of $m$ linearly independent vector fields on $M$. Then each commutator of pairs of vector fields can be expressed

$$[v_i, v_j] = \sum_{k=1}^{m} c_{ij}^k(q) v_k \tag{4}$$

Indeed, the coefficients are easily computed. Define

$$V = [v_1 \; v_2 \ldots v_m], \qquad U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = V^{-1}$$

$$\chi_{ij} = [c_{ij}^1 \; c_{ij}^2 \ldots c_{ij}^m]^t \tag{5}$$

Then Eq. (4) yields

$$\chi_{ij} = U[v_i, v_j] \quad \text{or} \quad c_{ij}^k = u_k[v_i, v_j] \tag{6}$$

Suppose $q(t) : [t_1, t_2] \rightarrow M$ is a smooth path; then $\dot{q}(t)$ denotes the tangent vector to the path at the point $q(t) \in M$. Thus, we can always express $\dot{q}$ as a linear combination of the tangent vectors $v_i$, $i = 1, \ldots, m$

$$\dot{q} = V(q)p \tag{7}$$

where

$$p = U(q)\dot{q} \tag{8}$$

The variables $p$ are called quasivelocities. Since these quantities are "velocities" we may associate them with a set of coordinates $\pi$, in the sense that $\dot{\pi} = p$. Although $\pi$ is well defined by this differential equation, its elements may not be true coordinates in the sense that there may not exist any function $\Pi(q)$ such that $\pi = \Pi(q)$. This follows from the observation that in view of Eq. (8) we must have

$$\delta\pi = U(q)\, dq \tag{9}$$

but the right-hand side (of each $\delta\pi_i$) may not be an exact differential—necessary conditions for the existance of $\Pi(q)$. Hence, the variables $\pi$ are referred to as quasicoordinates.

It is always possible to write the Lagrangian in terms of $q$ and $p$. Set $\tilde{\mathcal{L}}(q,p) = \mathcal{L}(q, \dot{q})$. In terms of $\tilde{\mathcal{L}}$ Lagrange's equations are attainable in the form given by the following lemma.

*Proposition 1:* Hamilton's principles leads to the equations of motion in terms of the coordinates $q, p$

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial p_k} - \sum_{i;j=1}^{m} c_{jk}^i \frac{\partial \tilde{\mathcal{L}}}{\partial p_i} p_j - v_k(\tilde{\mathcal{L}}) = Q^t v_k \tag{10}$$

or, equivalently,

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial p} - \sum_{j=1}^{m} p_j \frac{\partial \tilde{\mathcal{L}}}{\partial p} UX_j - \frac{\partial \tilde{\mathcal{L}}}{\partial q} V = Q^t V \tag{11}$$

where

$$v = [v_1 \; v_2 \; \ldots \; v_m]$$

and

$$X_j = \big[[v_j, v_1] \; [v_j, v_2] \ldots [v_j, v_m]\big]$$

A proof of the proposition as stated here is given in Ref. 7. Alternate derivations may be found in Refs. 8, 9, and 11.

*Remarks:* 1) In the literature these equations are referred to as Lagrange's equations in quasicoordinates[9] and also as Poincaré's equations, e.g., Refs. 7 and 8. Arnold et al.[7] attributes them to Poincaré circa 1901, and they were referred to as Poincaré's equations as early as 1941 by Chetaev.[14] They are related to Caplygin's equations, to the Boltzman-Hamel equations,[11] and also to the generalized Lagrange equations of Noble (see Ref. 15).

2) Poincaré's equations (11) along with Eq. (8) form a closed system of first-order differential equations that may be written in the form

$$\dot{q} = V(q)p \tag{12a}$$

$$\dot{p}^t \frac{\partial^2 \tilde{\mathcal{L}}}{\partial p^2} = -p^t V^t(q) \frac{\partial^2 \tilde{\mathcal{L}}}{\partial q^t \partial p} + \sum_{j=1}^{m} p_j \frac{\partial \tilde{\mathcal{L}}}{\partial p} UX_j + \frac{\partial \tilde{\mathcal{L}}}{\partial q} V + Q^t V \tag{12b}$$

3) In what follows, we describe the assembly of $V(q)$ and $\tilde{\mathcal{L}}(q,p)$ for articulated structures.

## III. Modeling of Articulated Spacecraft

Systematic methods for the formulation of the equations of motion for complex mechanical systems are now receiving considerable attention. Although there are important historical precedents, the investigations most relevant to us are those of Refs. 4–6 where certain recursive techniques for rigid body systems and also for systems with flexible elements have been formulated. In the following paragraphs we will review the necessary concepts and explain how they are integrated into the Lagrangian framework. The key issue is the formulation of the kinetic energy function, and we focus on that construction.

### A. Algorithmic Methods for Serial Chains of Rigid Bodies

We adopt the convention, by which any vector $a \in \Re^3$ is converted into a skew-symmetric matrix $\tilde{a}(a)$

$$\tilde{a}(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{13}$$

just as is commonly done for angular velocity.

Suppose $C$ is any point fixed in a rigid body. The spatial velocity at point $C$ of any body-fixed reference frame with origin at point $C$ is defined in Refs. 4–6 as $V_c = [\omega, v_c]$ where $v_c$ is the velocity of point $C$ and $\omega$ is the angular velocity of the body. Let $O$ be another point in the same body, and let $r_{co}$ denote the location of $C$ in the body frame with origin at $O$. Then the spatial velocity at point $C$ is related to that at $O$ by the relation

$$V_c = \phi(r_{co})V_0 \tag{14}$$

where

$$\phi(r_{co}) = \begin{bmatrix} I & 0 \\ -\tilde{r}_{co} & I \end{bmatrix} \tag{15a}$$

and its adjoint

$$\phi^*(r_{co}) = \begin{bmatrix} I & \tilde{r}_{co} \\ 0 & I \end{bmatrix} \tag{15b}$$

#### *1. Serial Chains of Rigid Bodies*

Now, let us consider a serial chain composed of $K+1$ rigid bodies connected by joints as illustrated in Fig. 1. The bodies are numbered 0 through $K$, with 0 denoting the base or reference body, which may represent any convenient inertial reference frame. The $k$th joint connects body $k-1$ at the point $C_{k-1}$ with body $k$ at the point $O_k$. Let $\mathfrak{F}^k$ denote a reference frame fixed in body $k$ with origin at $O_k$. The symbol $r_{co}^k$ denotes the vector from $O_k$ to $C_k$ in $\mathfrak{F}^k$, and $r^k$ denotes the vector from $O_k$ to $O_{k+1}$ in $\mathfrak{F}^k$. We will use a coordinate specific notation in which vectors represented in $\mathfrak{F}^i$ (or its tangent space) will be identified with a superscript $i$. Coordinate free relations carry no superscript. The $k$th joint has $n_k$, $1 \le n_k \le 6$ degrees of freedom, which can be characterized by $n_k$ quasivelocities $\beta(k)$ and a joint map matrix $H(k) \in \Re^{6\times n_k}$ so that $V_{o_k} - V_{c_{k-1}} = H(k)\beta(k)$.

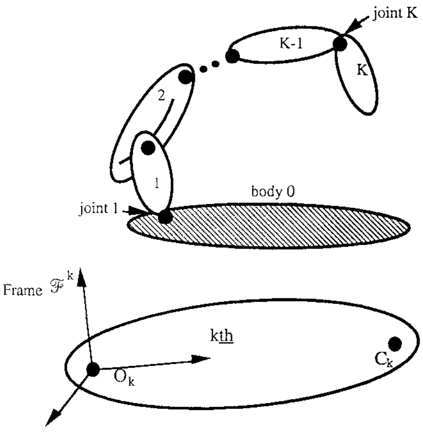


**Fig. 1 Serial chain composed of $K+1$ rigid bodies 0–$K$ and $K$ joints 1–$K$; on an arbitrary $k$th link the inboard and outboard joint hinge points are designated $O_k$ and $C_k$, the body fixed reference frame $\mathfrak{F}^k$ has its origin at $O_k$.**

Let the joint configuration parameters be denoted by $\sigma_k \in \Re^{n_k}$. Then the configuration rates $\dot{\sigma}_k$ are related to the quasivelocities by a relation

$$\dot{\sigma}_k = \Sigma_k(\sigma_k)\beta(k) \tag{16}$$

The joint rotation matrix that defines the relative orientation of $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$ can be realized as a function of $\sigma_k$, which we denote $L_{k,k-1}(\sigma_k)$.

Rodriguez,[4] Jain,[5] and Rodriguez and Jain[6] establish the recursive velocity relation, which we write in coordinate specific notation

$$V^i(k) = \phi\left[r_{co}^i(k-1)\right]V^i(k-1) + H^i(k)\beta^i(k) \tag{17}$$

Let us assume that $H(k)$ and $\beta(k)$ are specified in the frame $\mathfrak{F}^k$ and $V(k-1)$ has been computed in the frame $\mathfrak{F}^{k-1}$. Then it is convenient to compute $V(k)$ in the $k$th frame

$$V^k(k) = \text{diag}(L_{k-1,k}, L_{k-1,k})\phi\left[r_{co}^{k-1}(k-1)\right]V^{k-1}(k-1) + H^k(k)\beta^k(k) \tag{18}$$

If $V^0(0)$ is given, then Eq. (18) allows us to compute recursively, for $k = 1, \ldots, K$ the linear velocity of the origin of $\mathfrak{F}^k$ and the angular velocity of $\mathfrak{F}^k$, both represented in the coordinates of $\mathfrak{F}^k$. In what follows we take $V^0(0) = 0$. Abusing notation somewhat, it is convenient to define

$$\phi(k, k-1) = \text{diag}(L_{k-1,k}, L_{k-1,k})\phi\left[r_{co}^{k-1}(k-1)\right] \tag{19}$$

so that Eq. (18) can be written

$$V^k(k) = \phi(k, k-1)V^{k-1}(k-1) + H^k(k)\beta^k(k)$$
$$k = 1, \ldots, k, \qquad V^0(0) = 0 \tag{20}$$

It is necessary to define a spatial inertia tensor as well. Consider the $k$th rigid link and let $I_{\text{cm}}(k)$ denote the inertia tensor about the center of mass in coordinates $\mathfrak{F}^k$, $m(k)$ denote the mass, and $a(k)$ denote the position vector from the center of mass to an arbitrary point $O$. The spatial inertia about the center of mass $M_{\text{cm}}$ and about $O$, $M_o$ are

$$M_{\text{cm}}(k) = \begin{bmatrix} I_{\text{cm}} & 0 \\ 0 & mI \end{bmatrix}$$

$$M_o(k) = \phi^*(a)M_{\text{cm}}\phi(a) = \begin{bmatrix} I_0 & m\tilde{a} \\ -m\tilde{a} & mI \end{bmatrix} \tag{21}$$

where $I_o$ is the inertia tensor about $O$.

The spatial velocity and spatial inertia matrix and, hence, the kinetic energy function for the entire chain can now be conveniently constructed. Let us define the chain spatial velocity and joint quasivelocity

$$V = \left[V'(1), \ldots, V'(K)\right]', \qquad \beta = \left[\beta'(1), \ldots, \beta'(K)\right]' \tag{22}$$

so that we can write

$$V = \Phi H\beta \tag{23}$$

where

$$\Phi = \begin{bmatrix} I & 0 & \cdots & 0 \\ \phi(2,1) & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \phi(K,1) & \phi(K,2) & \cdots & I \end{bmatrix}$$

$$H = \begin{bmatrix} H(1) & 0 & \cdots & 0 \\ 0 & H(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & H(K) \end{bmatrix} \tag{24}$$

$$\phi(i,j) = \phi(i,i-1)\ldots\phi(j+1,j)$$

$$i = 2,\ldots,K \quad j = 1,\ldots,K-1$$

The kinetic energy function for the chain consisting of links 1–$K$ is

$$\text{K.E.}_{\text{chain}} = ½\beta'\mathfrak{M}\beta \tag{25}$$

where the chain inertia matrix is

$$\mathfrak{M} = H^*\Phi^*M\Phi H, \qquad M = \text{diag}\left[M_0(1),\ldots,M_0(K)\right] \tag{26}$$

*2. Sliding Joint-Rigid Body Case*

We define a sliding joint to be a one degree of freedom relative motion between two bodies involving translation along a path defined in one of the bodies. In general, this means that the joint map matrix may be a function of the joint configuration variables. Consider constrained relative motion between two bodies $k$ and $k-1$ in which body $k$ is free to translate along a path $\mathcal{P}$ defined in rigid body $k-1$. The path $\mathcal{P}$ can be characterized by a map $\gamma$: $\mathcal{R}\rightarrow\mathcal{R}^3$, so that $p = \{\gamma(\epsilon)|\,\epsilon_0 \le \epsilon \le \epsilon_1\}$. Specification of $\epsilon(t)$ defines the motion of a $p \in \mathcal{P}$ via the composite relation $\gamma[\epsilon(t)]$. It is convenient to think of the point $p$ as defining a (moving) point $C_{k-1}$ in body $k-1$ and a (fixed) point $O_k$ in body $k$. The relative velocity of a point $p$ moving along $p$ with respect to the body frame $\mathfrak{F}^{k-1}$ is

$$v_p^{k-1} = \frac{\partial\gamma}{\partial\epsilon}\dot{\epsilon} \tag{27}$$

The inertial velocity of $p$ as measured in $\mathfrak{F}^{k-1}$ is

$$v_p^{k-1} = v_o^{k-1}(k-1) - \tilde{\gamma}\omega^{k-1}(k-1) + \frac{\partial\gamma}{\partial\epsilon}\dot{\epsilon} \tag{28}$$

Suppose $\mathfrak{F}^k$ is a reference frame in body $k$ with origin at $O_k$ ($p$). We assume that the angular orientation of $\mathfrak{F}^k$ relative to $\mathfrak{F}^{k-1}$ remains constant with the corresponding axes of $\mathfrak{F}^k$ and $\mathfrak{F}^{k-1}$ aligned. This is only a matter of convenience because any joint involving both translation and rotation can be decomposed into two joints, one involving pure translation and a second involving pure rotation. We may take $\dot{\epsilon}$ to be the single translational quasivelocity $[\beta(k) = \dot{\epsilon}]$ so that the spatial velocity takes the form

$$V^k(k) = \phi\left[\gamma^{k-1}(\epsilon)\right]V^{k-1}(k-1) + H^k(k)\beta^k(k) \tag{29}$$

$$H^k(k) = \begin{bmatrix} 0_{3\times1} \\ \dfrac{\partial\gamma}{\partial\epsilon} \end{bmatrix}, \qquad \beta^k(k) = \dot{\epsilon} \tag{30}$$

As an example, suppose the path $p$ is a segment of the $z$ axis in $\mathfrak{F}^{k-1}$. Then

$$\gamma^{k-1} = \begin{bmatrix} 0 \\ 0 \\ \epsilon \end{bmatrix}, \qquad H^k(k) = \begin{bmatrix} 0_{5\times1} \\ 1 \end{bmatrix}$$

**B. Serial Chains of Flexible Bodies**

The procedure can be modified for flexible links. Let the reference frame $\mathfrak{F}^k$, with origin at $O_k$, be so oriented that its $z$ axis passes through $C_k$ in the undeformed configuration. We assume that each link is a one-dimensional, beam-like, flexible body and that the deformable centerline is coincident with the $z$ axis of $\mathfrak{F}^k$ in the undeformed configuration. The beam equations will be written in the frame $\mathfrak{F}^k$. Because the link is no longer a rigid body it does not make sense to have $\mathfrak{F}^k$ a body fixed frame. Instead, we attach $\mathfrak{F}^k$ to the body by requiring cantilever beam boundary conditions at $z=0$. In other words, $\mathfrak{F}^k$ may be thought of as fixed in an infinitesimal element at $O_k$. Figure 2 illustrates the $k$th body in the undeformed and deformed configurations. We assume that deformations are small. Let $F^k$ be a second reference frame with origin at $C_k$ and aligned with $\mathfrak{F}^k$ in the undeformed configuration. The orientation of $F^k$ under deformation is defined by fixing $F^k$ in an infinitesimal element at $C_k$, i.e., the location of its origin in $\mathfrak{F}^k$ is $\eta^k(z_c)$ and its relative angular orientation with respect to $\mathfrak{F}^k$ is $\xi^k(z_c)$.

Now, notice that

$$V_c^k(k) = \phi\left[\eta^k(z_c)\right]V_0^k(k) + \begin{bmatrix} \dot{\xi}(z_c) \\ \dot{\eta}(z_c) \end{bmatrix} \tag{31a}$$

$$V_0^k(k) = \text{diag}(L_{k-1}, L_{k-1,k})V_c^{k-1}(k-1) + H^k(k)\beta^k(k) \tag{31b}$$

where $L_{k,k-1}$ retains the meaning of the rigid body case, i.e., it is the rotation matrix that characterizes the relative orientation of the frame $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$. Likewise, the joint operator $H(k)$ and quasivelocity $\beta(k)$ retain their prior meaning. There is, however, a subtle distinction in the computation of $L_{k,k-1}$ in the present case vis-a-vis the rigid-body case. As before, the joint configuration variables are obtained from the differential equations

$$\dot{\sigma}_k = \Sigma_k(\sigma_k)\beta(k) \tag{32}$$

Then the rotation matrix is

$$L_{k,k-1} = L(\sigma_k)L\left[\xi(z_c)\right] \tag{33}$$

Suppose that a finite-dimensional model is obtained for each flexible body via finite element analysis (e.g., Ref. 3) so that the $k$th link is approximated with $N_k$ elements and, hence, $N_k+1$ nodes numbered $0, 1, \ldots, N_k$. We assume that node 0 coincides with the point $O$. Furthermore, each node may have as many as three displacements and three rotational degrees of freedom. Then the link deformations in $\mathfrak{F}^k$ are described by

$$\begin{bmatrix} \xi(z,t) \\ \eta(z,t) \end{bmatrix} = \begin{bmatrix} \bar{\xi}(t) \\ \bar{\eta}(t) \end{bmatrix}\Psi(z) \tag{34}$$

The relative (that is, with respect to the frame $\mathfrak{F}^k$) spatial velocity associated with an infinitesimal element located at an arbitrary $z$ in the undeformed configuration is

$$v(k,z) = \begin{bmatrix} \dot{\xi}(z,t) \\ \dot{\eta}(z,t) \end{bmatrix} = \begin{bmatrix} \dot{\bar{\xi}}(t) \\ \dot{\bar{\eta}}(t) \end{bmatrix}\Psi(z) = \left[v_1(k),\ldots,v_{N_k}(k)\right]\Psi(z) \tag{35}$$

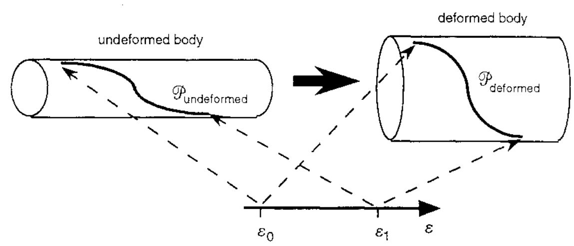


**Fig. 2 Characterization of the path $\mathcal{P}$ in terms of a parameter $\epsilon$. Since a deformation of the body deforms the path, $\mathcal{P}$ is defined by a map that depends on both the deformation (a function) and the parameter (a scalar).**

where the columns of $[v_1(k), \ldots, v_{N_k}(k)]$ are the nodal spatial velocities, in particular, at node $i$ ($z = z_i$) we use the notation

$$v_i(k) = \begin{bmatrix} \dot{\xi}(z_i, t) \\ \dot{\eta}(z_i, t) \end{bmatrix} = \begin{bmatrix} \dot{\xi}_i \\ \dot{\eta}_i \end{bmatrix}, \quad i = 1, \ldots, N_k, \quad v_0(k) = 0 \quad (36)$$

Now, we can combine Eqs. (31) and (35) to obtain the recursive formula

$$V_0^{k+1}(k+1) = \operatorname{diag}(L_{k,k+1}, L_{k,k+1})\Big\{\phi\big[\eta^k(z_c)\big] V_0^k(k) + v(k, z_c)\Big\} + H^{k+1}(k+1)\beta^{k+1}(k+1) \quad (37)$$

Since $V_o^0(0) = 0$, we can again compute all velocities $V_o^{k+1}(k+1)$ in terms of $v$ and $\beta$. It is convenient to again use stacked notation and define $v(k) = [v_1^t(k), \ldots, v_{N_k}^t(k)]^t$ and to adjoin to Eq. (37) the identity $v(k) = v(k)$. In this way, we construct the recursive formula

$$\begin{bmatrix} V_o(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} \phi(k+1,k) & \lambda(k+1,k) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_o(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} H(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \beta(k+1) \\ v(k+1) \end{bmatrix} \quad (38a)$$

where

$$\phi(k+1,k) = \operatorname{diag}(L_{k,k+1}, L_{k,k+1})\phi\big[\eta^k(z_c)\big] \quad (38b)$$

$$\lambda(k+1,k) = \operatorname{diag}(L_{k,k+1}, L_{k,k+1}) \times \big[\Psi_1(z_c) I_{6\times 6} \Psi_2 I_{6\times 6}, \ldots, \Psi_{N_k} I_{6\times 6}\big] \quad (38c)$$

Notice that if $z_c = z_i$, $i = 1, \ldots, N_k$, then

$$\lambda(k+1,k) = \operatorname{diag}(L_{k,k+1}, L_{k,k+1})[0_{6\times 6}, \ldots, \Psi_i I_{6\times 6}, \ldots, 0_{6\times 6}]$$

Thus, if point $C_k$ is fixed judiciously at one of the node points the matrix $\lambda$ simplifies. Finally, let us define the nodal spatial velocity vector $V(k)$ and the nodal quasivelocity vector $\pi(k)$ for the $k$th body

$$V(k) = \begin{bmatrix} V_0(k) \\ v(k) \end{bmatrix}, \qquad \pi(k) = \begin{bmatrix} \beta(k) \\ v(k) \end{bmatrix} \quad (39)$$

which allows us to write the recursion (38a) in the form.

$$V(k) = \phi(k, k-1)V(k-1) + H(k)\pi(k) \qquad k = 1, \ldots, K, \qquad V(0) = 0 \quad (40)$$

The similarity to the rigid case is obvious.

As in the rigid case, our goal is to construct the spatial velocity vector and the kinetic energy function for the entire chain. Let us define the chain spatial velocity and quasivelocity

$$V = \big[V^t(1), \ldots, V^t(K)\big]^t, \qquad \pi = \big[\pi^t(1), \ldots, \pi^t(K)\big]^t \quad (41)$$

so that we can write

$$V = \Phi H \pi \quad (42)$$

where

$$\Phi = \begin{bmatrix} I & 0 & \cdots & 0 \\ \phi(2,1) & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \phi(K,1) & \phi(K,2) & \cdots & I \end{bmatrix}$$

$$H = \begin{bmatrix} H(1) & 0 & \cdots & 0 \\ 0 & H(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & H(K) \end{bmatrix} \quad (43)$$

$$\phi(i,j) = \phi(i, i-1), \ldots, \phi(j+1, j) \qquad i = 2, \ldots, K \quad j = 1, \ldots, K-1$$

We assume that the kinetic energy for each link has been constructed (via finite element reduction) in the form

$$\text{K.E.}_{\text{link}\,k} = \tfrac{1}{2} V^t(k) M_0(k) V(k) \quad (44)$$

Then the kinetic energy function for the chain consisting of links 1-$K$ is

$$\text{K.E.}_{\text{chain}} = \tfrac{1}{2}\pi^t \mathfrak{M} \pi \quad (45)$$

where the chain inertia matrix is

$$\mathfrak{M} = H^* \Phi^* M \Phi H, \qquad M = \operatorname{diag}\big[M_o(1), \ldots, M_o(K)\big] \quad (46)$$

*1. Sliding Joints/Flexible-Body Case*

In a manner similar to the rigid case, we consider relative motion between two bodies $k$ and $k-1$ in which body $k$ translates along a path $\mathcal{P}$ defined in body $k-1$. Let $X$ denote the function space of deformations of body $k-1$. Then the path $\mathcal{P}$ may be characterized by a map $\gamma: X \times \mathcal{R} \to \mathcal{R}^3$, so that $\mathcal{P} = \{\gamma(\eta, \xi, \epsilon) \mid \epsilon_0 \le \epsilon \le \epsilon_1 \text{ and } [\eta(z,t), \xi(z,t)] \in X\}$, see Fig. 2. In the undeformed configuration the path $\mathcal{P}$ is defined by the map $\gamma(0, 0, \epsilon)$. When the deformations are approximated by a finite element model, as described earlier, then the path $\mathcal{P}$ is approximated by a map $\hat{\gamma}(\bar{\eta}, \bar{\xi}, \epsilon)$. Once again, we consider a reference frame fixed in body $k$, $\mathfrak{F}^k$, with origin moving with the point $p$. Unlike the rigid case, it is convenient to specify the relative orientation of $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$ in a more general way. The relative angular alignment of $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$ is defined in terms of the location of $p$ and the deformations via a function $\varphi(\eta, \xi, \epsilon)$ or its approximation $\hat{\varphi}(\bar{\eta}, \bar{\xi}, \epsilon)$.

The relative velocity of a point $p$ traversing the path with respect to the body frame $\mathfrak{F}^{k-1}$ is

$$v_p(k-1) = \dot{p} = \frac{d\hat{\gamma}}{dt} = \frac{\partial \hat{\gamma}}{\partial \bar{\eta}}\dot{\bar{\eta}}(t) + \frac{\partial \hat{\gamma}}{\partial \bar{\xi}}\dot{\bar{\xi}}(t) + \frac{\partial \hat{\gamma}}{\partial \epsilon}\dot{\epsilon}(t) \quad (47)$$

The inertial velocity of $p$ as measured in $\mathfrak{F}^{k-1}$ is

$$v_p^{k-1} = v_o^{k-1}(k-1) - \tilde{\hat{\gamma}}\omega^{k-1}(k-1) + \frac{\partial \hat{\gamma}}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{\partial \hat{\gamma}}{\partial \bar{\xi}}\dot{\bar{\xi}} + \frac{\partial \hat{\gamma}}{\partial \epsilon}\dot{\epsilon} \quad (48)$$

Let us define

$$\Gamma(k-1) = \big[\Gamma_1, \ldots, \Gamma_{N_{k-1}}\big], \quad \text{where} \quad \Gamma_i = \begin{bmatrix} \dfrac{\partial \hat{\gamma}}{\partial \bar{\xi}_i} & \dfrac{\partial \hat{\gamma}}{\partial \bar{\eta}_i} \end{bmatrix} \quad (49)$$

so that

$$v_p^{k-1} = v_o^{k-1}(k-1) - \tilde{\hat{\gamma}}\omega^{k-1}(k-1) + \Gamma(k-1)v(k-1) + \frac{\partial \hat{\gamma}}{\partial \epsilon}\dot{\epsilon} \quad (50)$$

If we assume small deformations the relative angular velocity of $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$ as measured in $\mathfrak{F}^{k-1}$ is

$$\omega_p^{k-1} = \frac{d\hat{\varphi}}{dt} = \frac{\partial \hat{\varphi}}{\partial \bar{\eta}}\dot{\bar{\eta}}(t) + \frac{\partial \hat{\varphi}}{\partial \bar{\xi}}\dot{\bar{\xi}}(t) + \frac{\partial \hat{\varphi}}{\partial \epsilon}\dot{\epsilon}(t) \quad (51)$$

Let us define

$$\Lambda(k-1) = \big[\Lambda_1 \ldots \Lambda_{N_{k-1}}\big], \quad \text{where} \quad \Lambda_i = \begin{bmatrix} \dfrac{\partial \hat{\varphi}}{\partial \bar{\xi}_i} & \dfrac{\partial \hat{\varphi}}{\partial \bar{\eta}_i} \end{bmatrix} \quad (52)$$

so that

$$\omega_p^{k-1} = \Lambda(k-1)v(k-1) + \frac{\partial \hat{\varphi}}{\partial \epsilon}\dot{\epsilon}(t) \quad (53)$$

We may take $\dot\epsilon$ to be the single translational quasivelocity $[\beta(k)=\dot\epsilon]$ so that the spatial velocity takes the form

$$V_o^k(k) = \mathrm{diag}(L_{k-1,k}, L_{k-1,k})\left\{\phi\left[\hat\gamma(\bar\eta, \bar\xi, \epsilon)\right] V_o^{k-1}(k) + \begin{bmatrix}\Lambda(k-1)\\ \Gamma(k-1)\end{bmatrix} v(k-1)\right\} + H^k(k)\beta^k(k) \tag{54}$$

$$H^k(k) = \mathrm{diag}(L_{k-1,k}, L_{k-1,k})\begin{bmatrix}\dfrac{\partial\hat\varphi}{\partial\epsilon}\\ \dfrac{\partial\hat\gamma}{\partial\epsilon}\end{bmatrix}, \qquad \beta^k(k) = \dot\epsilon \tag{55}$$

To complete the correlation with Eq. (38a), we need only identify

$$\begin{aligned}\phi(k, k-1) &= \mathrm{diag}(L_{k-1,k}, L_{k-1,k})\phi\left(\hat\gamma(\bar\eta, \bar\xi, \epsilon)\right)\\ \lambda(k, k-1) &= \mathrm{diag}(L_{k-1,k}, L_{k-1,k})\begin{bmatrix}\Lambda(k-1)\\ \Gamma(k-1)\end{bmatrix}\end{aligned} \tag{56}$$

As an example, suppose the path $\mathcal{P}$ corresponds to a segment of the $z$ axis in $\mathfrak{F}^{k-1}$ so that

$$\hat\gamma(\bar\eta, \bar\xi, \epsilon) = \bar\eta(t)\Psi(\epsilon)$$

In addition, suppose that the relative angular orientation of $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$ is precisely the angular deformation of body $k-1$ at $p$

$$\hat\varphi(\bar\eta, \bar\xi, \epsilon) = \bar\xi(t)\Psi(\epsilon)$$

The relative linear and angular velocities of $\mathfrak{F}^k$ with respect to $\mathfrak{F}^{k-1}$ are (represented in the $\mathfrak{F}^{k-1}$ frame)

$$v_p^{k-1} = \dot{\bar\eta}(t)\Psi(\epsilon) + \bar\eta(t)\frac{\partial\Psi(\epsilon)}{\partial\epsilon}\dot\epsilon$$

$$\omega_p^{k-1} = \dot{\bar\xi}(t)\Psi(\epsilon) + \bar\xi(t)\frac{\partial\Psi(\epsilon)}{\partial\epsilon}\dot\epsilon$$

If longitudinal deformations are neglected, i.e., $\eta_3 = z$, we obtain the further reduction

$$\hat\gamma(\bar\eta, \bar\xi, \epsilon) = \begin{bmatrix}\bar\eta_1\Psi(\epsilon)\\ \bar\eta_2\Psi(\epsilon)\\ \epsilon\end{bmatrix}, \quad v_p^{k-1} = \begin{bmatrix}\dot{\bar\eta}_1\Psi(\epsilon)\\ \dot{\bar\eta}_2\Psi(\epsilon)\\ 0\end{bmatrix} + \begin{bmatrix}\bar\eta_1\dfrac{d\Psi(\epsilon)}{d\epsilon}\\ \bar\eta_2\dfrac{d\Psi(\epsilon)}{d\epsilon}\\ 1\end{bmatrix}\dot\epsilon$$

#### 2. Remark on Finite Element Reduction

One approach to finite element reduction is based on collocation by splines. Our implementation of this method is described in Ref. 3. It is simple and convenient for the class of models of interest here.

#### 3. Remark on the Structure of Poincaré's Equations

The preceding definitions and constructions provide the kinetic energy function in the form $\mathfrak{I}(q,p) = p^t\mathfrak{M}(q)p$ ($\pi$ replaces $p$ in the flexible case). Hence, we reduce Eq. (12b) to the form

$$\mathfrak{M}(q)\dot p + \mathcal{C}(q,p)p + \mathfrak{F}(q) = Q_p \tag{57a}$$

where

$$\mathcal{C}(q,p) = -\left[\frac{\partial(\mathfrak{M}p)}{\partial q}V\right]p + \frac{1}{2}\left[\frac{\partial(\mathfrak{M}p)}{\partial q}V\right]^t p + \sum_{j=1}^{m} p_j X_j^t U^t \mathfrak{M} \tag{57b}$$

$$\mathfrak{F}(q) = V^t(q)\frac{\partial\mathcal{V}(q)}{\partial q^t}, \qquad Q_p = V^t(q)Q \tag{57c}$$

Notice that $Q_p$ denotes the generalized forces represented in the $p$-coordinate frame whereas $Q$ denotes the generalized forces in the $\dot q$-coordinate frame (aligned with $q$). $Q_p$ is actually more convenient because the quasivelocities are usually represented in appropriate body frames.

#### 4. Remark on Taylor Linearization

If $Q_p$ is constant, it makes sense to discuss equilibria of the system defined by Eqs. (12a) and (57a). An equilibrium point is defined as a value of the state $(q,p)$ such that $\dot q = 0$ and $\dot p = 0$. From Eq. (12a) and the invertibility of $V(q)$ we find that $p = 0$ at an equilibrium point. An equilibrium value of $q$ then satisfies $\mathfrak{F}(q) = Q_p$. For convenience, let the equilibrium point of interest correspond to $q = 0$. A straightforward computation shows that the Taylor linearized dynamics are

$$\dot q = V(0)p \tag{58a}$$

$$\mathfrak{M}(0)\dot p + \mathcal{C}(0,0)p + \frac{\partial\mathfrak{F}}{\partial q}(0)q = \Delta Q_p \tag{58b}$$

## IV. Nonlinear Attitude Control via Partial Feedback Linearizing

The approach to attitude control design considered here derives from a well-established theoretical basis for control design by feedback linearization.[16] In recent work, including Ref. 3, we have tailored this technique to take advantage of the special structure of Lagrangian dynamics.

### A. Partial Feedback Linearizing Control

The spacecraft models just formulated reduce to dynamical equations of the form

$$\dot q = V(q)p \tag{59a}$$

$$\mathfrak{M}(q,t)\dot p + \mathcal{C}(q,p,t)p + \mathfrak{F}(q,t) = G\tau \tag{59b}$$

where $q$ are a set of generalized coordinates and $p$ a set of quasivelocities. The class of attitude control problems we investigate is best characterized by partitioning the coordinate vector, and correspondingly the quasivelocity vector, into two parts

$$q = \begin{bmatrix}\xi\\ u\end{bmatrix}, \qquad p = \begin{bmatrix}\omega\\ v\end{bmatrix} \tag{60}$$

where $\xi$ represents the controlled body attitude parameters and $\omega$ the corresponding body angular velocity, whereas $u$ and $v$ represent the remaining coordinates and velocities, respectively. Then in partitioned form, the equations are

$$\dot\xi = \Gamma(\xi)\omega \tag{61a}$$

$$\dot u = \Sigma(\xi,u)v \tag{61b}$$

$$M_\omega\dot\omega + N\dot v + F_\omega = G_\omega\tau \tag{61c}$$

$$N^T\dot\omega + M_v\dot v + F_v = G_v\tau \tag{61d}$$

Our goal is to regulate the outputs $y = \xi$. The concept of partial feedback linearization (PFL) is a general approach to the design of nonlinear control systems for a general class of systems with smooth nonlinearities.[16] Attitude control of spacecraft using feedback linearization was first used by Dwyer.[17] A PFL compensation for the system (61) is a nonlinear feedback law of the form

$$\tau = \mathcal{A}(\xi,\omega,u,v,t) + \mathcal{B}(\xi,\omega,u,v,t)\alpha \tag{62}$$

which provides a closed-loop attitude response in the linear, decoupled form

$$\ddot{\xi} = \alpha \tag{63}$$

Specific conditions for the existence and construction of such controllers are given in Isidori.[16] Here we extend the discussion in Ref. 3 on the construction of PFL controllers for spacecraft modeled by Poincaré's equations.

The main constructive result is summarized in the following proposition.

*Proposition 1:* The PFL control for regulation of the outputs $y = \xi$ for the system defined by Eq. (61) takes the form of Eq. (62) with

$$\mathcal{A} = \left[G_\omega - NM_v^{-1}G_v\right]^{-1}\Big\{F_\omega - NM_v^{-1}F_v + \left[NM_v^{-1}N^T - M_\omega\right]\Gamma^{-1}\frac{\partial \Gamma\omega}{\partial \xi}\Gamma\omega\Big\}$$

$$\mathcal{B} = \left[G_\omega - NM_v^{-1}G_v\right]^{-1}\left[M_\omega - NM_v^{-1}N^T\right]\Gamma^{-1} \tag{64}$$

*Proof:* We prove the proposition by direct construction, in two steps. First, we use linearizing feedback to reduce Eq. (61c) to the form $\dot{\omega} = \beta$, which we then reduce to Eq. (63) by a second linearizing feedback. The composition of these to control laws gives the desired result. Equation (61d) can be solved for $\dot{v}$

$$\dot{v} = -M_v^{-1}N^T\dot{\omega} - M_v^{-1}F_v + M_v^{-1}G_v\tau$$

which allows its elimination from Eq. (61c)

$$\left[M_\omega - NM_v^{-1}N^T\right]\dot{\omega} + F_\omega - NM_v^{-1}F_v = \left[G_\omega - NM_v^{-1}G_v\right]\tau$$

Now we choose the feedback control law

$$\tau = \left[G_\omega - NM_v^{-1}G_v\right]^{-1}\Big\{F_\omega - NM_v^{-1}F_v + \left[M_\omega - NM_v^{-1}N^T\right]\beta\Big\}$$

which yields

$$\dot{\omega} = \beta$$

Now, differentiation of Eq. (61a) provides

$$\ddot{\xi} = \frac{\partial \Gamma\omega}{\partial \xi}\dot{\xi} + \Gamma(\xi)\dot{\omega} = \frac{\partial \Gamma\omega}{\partial \xi}\Gamma(\xi)\omega + \Gamma(\xi)\beta$$

Choose the control law

$$\beta = \Gamma^{-1}(\xi)\Big\{\alpha - \frac{\partial \Gamma\omega}{\partial \xi}\Gamma(\xi)\omega\Big\}$$

to obtain

$$\ddot{\xi} = \alpha$$

and the desired composite linearizing control law is

$$\tau = \left[G_\omega - NM_v^{-1}G_v\right]^{-1}\Big\{F_\omega - NM_v^{-1}F_v + \left[M_\omega - NM_v^{-1}N^T\right]\Gamma^{-1}\left[\alpha - \frac{\partial \Gamma\omega}{\partial \xi}\Gamma\omega\right]\Big\}$$

which is the stated result. §

*Remarks:* 1) The linearizing control law is local if the parameterization of the angular configuration is local. However, there is some flexibility here because one may choose alternate parameterizations (e.g., Gibbs or Euler parameters), as appropriate to the problem.

2) The zero dynamics[16] are obtained by straightforward calculation

$$M_v\dot{v} + F_v - G_v\left[G_\omega - NM_v^{-1}G_v\right]^{-1}\left\{F_\omega - NM_v^{-1}F_v\right\} = 0$$

with $\xi = 0$ and $\omega = 0$.

3) In the specific problem of interest here we have $G_\omega = I_3$ and $G_v = 0$, so that Eq. (64) simplifies somewhat to

$$\mathcal{A} = \Big\{F_\omega - NM_v^{-1}F_v + \left[NM_v^{-1}N^T - M_\omega\right]\Gamma^{-1}\frac{\partial \Gamma\omega}{\partial \xi}\Gamma\omega\Big\} \tag{65a}$$

$$\mathcal{B} = \left[M_\omega - NM_v^{-1}N^T\right]\Gamma^{-1} \tag{65b}$$

In this case the zero dynamics reduce to

$$M_v\dot{v} + F_v = 0$$

with $\xi = 0$ and $\omega = 0$.

4) The invertibility of $M_v$ is assured because it is an inertia matrix for a physical subsystem and is consequently a positive definite matrix.

5) If Gibbs parameters are used for attitude parameterization, then the last term in Eq. (64a) or (65a) simplifies because

$$\Gamma^{-1}\frac{\partial \Gamma\omega}{\partial x}\Gamma\omega = \gamma^T\omega\omega$$

where $\gamma$ is the vector of Gibbs parameters.

6) Equation (63) may be rewritten

$$\dot{z} = Az + B\alpha, \qquad A = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ I_3 \end{bmatrix} \tag{66a}$$

we may easily choose a stabilizing control for Eq. (63)

$$\alpha = K_p\xi + K_r\dot{\xi} = Kz \tag{66b}$$

### B. Adaptive Partial Feedback Linearizing Control

Because feedback linearization is a model-based approach to control system design, it is necessary to anticipate some sensitivity to model uncertainty. In the present case, it is reasonable to assume that the kinematics are precisely known but that the dynamics are not. Thus, we consider the situation where the model contains uncertain parameters, denoted $\Theta$, which belong to a bounded set $\mathcal{J}$. Equations (61) may be rewritten with these parameters explicitly shown

$$M_\omega(\Theta)\dot{\omega} + N(\Theta)\dot{v} + F_\omega(\Theta) = G_\omega\tau \tag{67a}$$

$$N(\Theta)^T\dot{\omega} + M_v(\Theta)\dot{v} + F_v(\Theta) = G_v\tau \tag{67b}$$

Because of its physical meaning, the invertibility of $M_v(\Theta)$ is preserved for all values of $\Theta \in \mathcal{J}$. Consequently, a feedback linearizing control exists for all parameter values. Indeed, the control (62) as constructed via proposition 1 is a parameter dependent control, which we rewrite in the form

$$\tau(\Theta) = \mathcal{A}_\Theta(\xi,\omega,u,v,t) + \mathcal{B}_\Theta(\xi,\omega,u,v,t)\alpha \tag{68}$$

The idea is to implement Eq. (68) with $\mathcal{J}$ replaced by an estimate $\hat{\mathcal{J}}$. When the estimated control $\tau(\hat{\mathcal{J}})$ is applied, the system

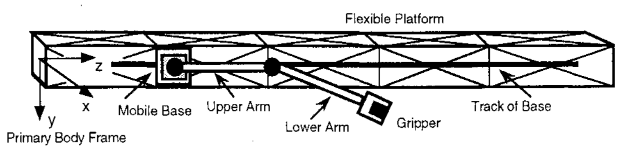


**Fig. 3 Considered system composed of a flexible platform, a mobile base, and the flexible upper and lower arms; for the purposes of the present study the gripper and payload are assumed fixed to the lower arm, i.e., the "wrist" is locked.**

is not exactly feedback linearized, and a simple computation shows that Eq. (63) is replaced by

$$\ddot{\xi} = \alpha + \Delta(\hat{\Theta},\Theta,\xi,\omega,u,v,t) \quad (69)$$

The following proposition provides a parameter adaptive feedback linearizing control law.

*Proposition 2:* Consider the system defined by Eq. (61) and with control $\tau(\hat{\vartheta})$ where $\tau(\cdot)$ is given by Eq. (68) and $\alpha$ by Eq. (66b). Suppose that the residual $\Delta$ defined in Eq. (69) has the form

$$\Delta(\hat{\Theta},\Theta,\xi,\omega,u,v,t) = \Psi(\xi,\omega,u,v,t)(\Theta-\hat{\Theta}) \quad (70)$$

Then asymptotic attitude stabilization ($\xi,\omega \rightarrow 0$) is achieved with the parameter estimator

$$\dot{\Theta} = Q\Psi^t(\xi,\omega,u,v,t)B^tPz \quad (71)$$

where $P$ is a symmetric, positive definite solution of

$$(A+BK)^t + P(A+BK) = -I \quad (72)$$

and $Q$ is any symmetric, positive definite marix. Various forms of this result are well known, e.g., Ref. 18.

## V. Summary of Simulation Results for Space Station Freedom/Mobile Remote Manipulator System

In the following paragraphs we describe simulation results that compare linear and nonlinear (PFL) controllers for attitude control of a prototype space station. Prior to consideration of a flexible platform, studies were conducted with a rigid platform. Although not reported here occasional comparative remarks will be made to the rigid case. The flexible platform case is far more complex than the rigid case in three respects: 1) the dynamical equations of motion are more involved, and this complexity increases the simulation times substantially; 2) the numerical analysis is much more subtle because there is substantial time scale separation due to fast platform vibration dynamics; and 3) the system dynamics are far more complicated with nonlinear inertial couplings between flexible and rigid body dynamics profoundly affecting system behavior.

### A. System Configuration

The space station with MRMS is idealized to be composed of four articulated elements: the space station main body (body 1), the MRMS base (body 2), the upper (inner) MRMS arm (body 3), and the lower (outer) MRMS arm (body 4). It is assumed that the MRMS base, body 2, can move along a fixed path on the space station, body 1, whereas body 3 is joined to body 2 and body 4 to body 3 via joints with up to three rotational degrees of freedom. The setup is illustrated in Fig. 3. We develop the explicit model for the case where the MRMS joints are restricted to one degree of freedom. Suppose that joint 3 admits only rotations about the $z$ axis in the $\mathfrak{F}^3$ frame and joint 4 about the $x$ axis in the $\mathfrak{F}^4$ frame.

The configuration variables are as follows.

1) $R \in \mathbb{R}^3$, the location of point $O_1$ on body 1 relative to inertial space.

2) $L_1 \in SO(3)$, the relative angular orientation of $\mathfrak{F}^1$ with respect to inertial space.

3) $\xi_i \in \mathbb{R}^3$, $\bar{\eta}_i \in \mathbb{R}^2$, $i=1,\ldots,N(=2)$ platform angular and displacement deformation coordinates.

4) $\zeta \in \mathbb{R}$, the location of the MRMS base along the undeformed track in the frame $\mathfrak{F}^1$.

5) $\psi_{32} \in \mathbb{R}$, the relative angular orientation of $\mathfrak{F}^3$ with respect to $\mathfrak{F}^2$.

6) $\phi_{43} \in \mathbb{R}$, the relative angular orientation of $\mathfrak{F}^4$ with respect to $\mathfrak{F}^3$.

The joint quasivelocities are $\beta(1)=(\omega_1,v_1)$ the linear velocity $v_1$ and the angular velocity $\omega_1$ of $\mathfrak{F}^1$; the linear velocity $\beta(2)=v_{2z}$ for joint 2; and the relative angular velocities $\beta(3)=\omega_{32}$ and $\beta(4)=\omega_{43}$ for joints 3 and 4.

Table 1 provides the physical data used in the simulation studies. This data is adapted from Ref. 1. Since only the weight and length of the beam have been specified, we assume the geometry and the material properties as follows. The geometry of the beam is a uniform, square boxbeam with outside dimension of 5 m. Material properties are density $\rho=7.860\times10^3$ kg/m$^3$, modulus of elasticity $E=200\times10^8$ N/m$^2$, and shear modulus $G=79\times10^8$ N/m$^2$. All of the platform characteristics except dissipation properties follow from these assumptions. A material dissipation model of the type described in Ref. 3 is assumed. In addition, we assume some form of active or passive vibration suppression provides additional damping. Even so, the dominant modes of the structure are very lightly damped as will be seen in the simulation results.

The beam model is developed in accordance with the finite element method described in Ref. 3, using collocation by splines as applied to a Timoshenko formulation of beam dynamics precisely as employed for the SCOLE mast. Using two elements, the resultant model provides 10 beam flexure degrees of freedom and hence 20 beam modes, most of which are of very high frequency and well out of the control bandwidth. The resultant stiff system requires simulation times excessive for analysis. Thus, we reduce the system to retain 4 beam-flexure degrees of freedom and consequently eight modes by retaining the so-called long wavelength dynamics, i.e., we eliminate the angular deformation degrees of freedom. Of these, four modes are near the control bandwidth (natural frequencies of about 3 rad/s) and the others are outside the bandwidth (approximately 10 rad/s). For convenience, we define the deformation coordinate vector $\bar{\eta}=[\bar{\eta}_1{}^t,\bar{\eta}_2{}^t]^t \in \mathbb{R}^4$. A few comparative simulations with 20 and 8 modes provided no observable effect.

**Table 1 Physical data adapted from Ref. 1**

| Body | Length, m | Mass, kg | Inertia, kg-m$^2$ | cm location, m |
|---|---|---|---|---|
| Space station | 110 | 211,258 | $J_x=2.13\times10^8$ | $x=0$ |
| | | | $J_y=2.13\times10^8$ | $y=0$ |
| | | | $J_z=880{,}214.6$ | $z=55$ |
| Mobile base | 1.5 | 316.9 | $J_x=178.25$ | $x=0$ |
| | | | $J_y=178.25$ | $y=0$ |
| | | | $J_z=356.5$ | $z=0$ |
| Upper arm | 14.3 | 3169 | $J_x=54{,}002$ | $x=0$ |
| | | | $J_y=54{,}002$ | $y=0$ |
| | | | $J_z=0$ | $z=7.15$ |
| Lower arm | 14.3 | 3169 | $J_x=54{,}002$ | $x=0$ |
| | | | $J_y=54{,}002$ | $y=0$ |
| | | | $J_z=0$ | $z=7.15$ |

### B. System Equations

The dynamical equations of motion for the composite system including the space station with MRMS have been derived in terms of Poincaré's equations and take the form

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{R} \\ \dot{\eta} \\ \dot{\zeta} \\ \dot{\psi}_{32} \\ \dot{\phi}_{43} \end{bmatrix} = \begin{bmatrix} \Gamma(\xi_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & L_1(\xi_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{4\times 4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ v_1 \\ v \\ v_{2z} \\ \omega_{32z} \\ \omega_{43x} \end{bmatrix} \tag{73a}$$

$$M\dot{p} - \left[\frac{\partial(Mp)}{\partial q} V\right] p + \frac{1}{2}\left[\frac{\partial(Mp)}{\partial q} V\right]^t p + \begin{bmatrix} \tilde{\omega}_1 & \tilde{v}_1 & 0 \\ 0 & \tilde{\omega}_1 & 0 \\ 0 & 0 & 0_{7\times 7} \end{bmatrix} Mp - \begin{bmatrix} 0_6 & 0 & 0 \\ 0 & B_s & 0 \\ 0 & 0 & 0_3 \end{bmatrix} p - \begin{bmatrix} 0_6 & 0 & 0 \\ 0 & K_s & 0 \\ 0 & 0 & 0_3 \end{bmatrix} q = Q_p \tag{73b}$$

In this study we prescribe the MRMS motion and determine the corresponding SSF response. The MRMS motion is defined by prescribing the MRMS acceleration and computing the resultant motion using the kinematics Eq. (73a). Thus, we have

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\psi}_{32} \\ \dot{\phi}_{43} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{2z} \\ \omega_{32z} \\ \omega_{43x} \end{bmatrix} \tag{74a}$$

$$\begin{bmatrix} \dot{v}_{2z} \\ \dot{\omega}_{32z} \\ \dot{\omega}_{43x} \end{bmatrix} = \begin{bmatrix} a_{2z} \\ a_{32z} \\ a_{43x} \end{bmatrix} \tag{74b}$$

In all of the subsequent simulations we use the above MRMS motion model with the accelerations $a_{2z}$, $a_{32z}$, $a_{43x}$ prescribed as constants. There remains a great deal of flexibility in this model because in addition to specifying the accelerations, the initial conditions on velocities and configuration variables may also be prescribed. With the motion of the MRMS prescribed, the equations governing the response of the space station are obtained by stripping off the first three equations of Eqs. (73a) and (73b).

### C. Simulation Results

We briefly summarize our simulation experience with the linear regulator, the PFL regulator, and the adaptive PFL regulator.

#### *1. Stabilization with Linear Feedback*

The linear attitude regulator was designed as a decoupling controller so that meaningful comparisons can be made with the PFL designs. Table 2 lists the open- and closed-loop eigenvalues for several different feedback gain values. The open-loop set consists of 12 zero eigenvalues corresponding to the rigid body dynamics and an additional eight corresponding to the platform flexure dynamics. The second column lists the eigenvalues resulting from a design intended to achieve the same attitude response as had been achieved in a study of the rigid body case. Notice that the first 14 eigenvalues correspond to the "zero dynamics" and remain fixed as the attitude gain is "detuned" in columns three and four. The zero dynamics modes include the six rigid body translation modes and eight cantilevered beam modes of the platform. Although the nominal closed-loop linear system is stable, application of the linear regulator to the nonlinear simulation with 0.1 rad error in each Euler angle yields a divergent trajectory. This is due to destabilizing inertial crosscoupling between the flexible and rigid body dynamics. Detuning of the closed loop appeared appropriate to reduce slewing rates and hence platform flexure. Moreover, it is clear that MRMS motion and attitude regulation performance will not in practice approach the levels demanded here. For example, we impose an MRMS translation of 18 m in 60 s, whereas, Wie et al.[2] impose a translation of 5 m in 300 s.

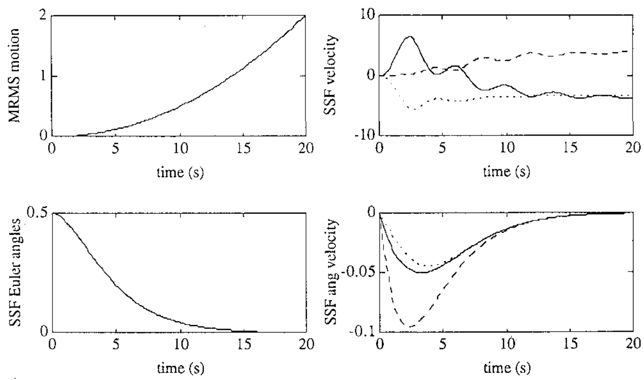


**Fig. 4 Effectiveness of combined PFL decoupling and attitude stabilization: the MRMS motion illustrates three curves that overlay each other, with base translation (m) and two joint angles (rad); similarly all three Euler angles follow the same trajectory (rad). Notice that platform flexure is completely decoupled from the attitude response although its effects are visible in the uncontrolled translational velocities (m/s).**

**Table 2 Open- and closed-loop eigenvalues**

| Open loop | Nominal closed loop, $k$ | Detuned closed loop, $k/8$ | |
|---|---|---|---|
| $0^2$ | $0^2$ | $0^2$ | Translation modes: |
| $0^2$ | $0^2$ | $0^2$ | unaffected by control torque |
| $0^2$ | $0^2$ | $0^2$ | |
| $-10.4212 \pm 10.5963i$ | $-0.1763 \pm 3.3205i$ | $-0.1763 \pm 3.3205i$ | Flexure modes: |
| $-10.8762 \pm 10.5876i$ | $-0.1763 \pm 3.3205i$ | $-0.1763 \pm 3.3205i$ | correspond to zero dynamics |
| $-0.2053 \pm 3.3267i$ | $-0.1364 \pm 1.6505i$ | $-0.1364 \pm 1.6505i$ | in closed loop |
| $-0.2053 \pm 3.3290i$ | $-0.1364 \pm 1.6505i$ | $-0.1364 \pm 1.6505i$ | |
| $0^2$ | $-0.4, -0.4$ | $-0.05, -0.05$ | Rotation modes: |
| $0^2$ | $-0.4, -0.4$ | $-0.05, -0.05$ | attitude is stabilized |
| $0^2$ | $-0.4, -0.4$ | $-0.05, -0.05$ | by feedback |

Nevertheless, the detuned regulators still produce divergent trajectories, although they are somewhat less dramatic. The trajectories corresponding to the last column, i.e., the least aggressive design, are also divergent. Reduction of the initial attitude errors to 0.01 rad, however, provides convergent trajectories. We can conclude that the anticipated stable linear behavior is indeed observed in very small signal excursions. The significance of the nonlinear interactions that arise through the inertial couplings is quite striking. It is anticipated that further detuning would lead to a larger domain of attraction for the stable equilibrium point, although we have not confirmed this. Even so, it is clear that the achievable performance with linear regulators is severely limited.

*2. Decoupling and Stabilization via Partial Feedback Linearization*

We first consider attitude regulation with an MRMS maneuver combined with initial attitude errors and with perfect knowledge of all parameters. The PFL control results are illustrated in Fig. 4.

*3. Parameter Uncertainty and Adaptive Partial Feedback Linearization*

We begin by considering the effect of a 5% stiffness uncertainty on the performance of the decoupling and stabilizing PFL controller. Simulation results show that attitude regulation is seriously degraded even with this rather minimal uncertainty. This sensitivity is consistent with our prior observations about the linear regulator and, in fact, it is likely that sensitivity would be substantially reduced by detuning of the stabilizer and reduction of the rate of MRMS motion.

The addition of adaptation to the PFL controller restores its excellent performance even with 10% stiffness uncertainty. Somewhat less satisfactory results have been achieved with 15% uncertainty. However, 20% uncertainty results in serious degradation of performance.

## VI. Conclusions

This paper summarizes results of a study of the application of partial feedback linearization methods to the attitude control of an articulated spacecraft representative of the Space Station Freedom with a mobile remote manipulator system (MRMS). Computer studies contrast linear state feedback attitude stabilizers with partial feedback linearization- (PFL) based attitude stabilizers. The results presented here confirm previous observations that MRMS motion can significantly degrade and even destabilize attitude regulation when linear controllers are applied to this highly nonlinear dynamical system. Our results show that in the flexible case the linear regulator must be significantly detuned to achieve stable responses. In fact, even with detuning, the attitude errors must be very small to observe the behavior predicted by linear theory. Parameter uncertainty is not tolerable. Although the studies conducted to date are far from exhausting, it is clear that PFL design is promising. It is shown that the PFL controller performs quite well with perfect knowledge (no parameter uncertainty) both with respect to decoupling and stabilization. However, performance deteriorates rapidly with even small parametric uncertainties. Adaptive PFL is shown to restore the PFL performance with uncertainties of 10%. Controller detuning will certainly improve robustness, and studies that address the tradeoff between performance and sensitivity would be required in any given design situation.

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## References

[1]Mah, H. W., Modi, V. J., Morita, Y., and Yokoto, H., "Dynamics and Control During Slewing Maneuvers," *Acta Automatica,* Vol. 19, No. 2, 1989, pp. 125–143.

[2]Wie, B., Hunt, A., and Singh, R., "Multibody Interaction Effects on Space Station Attitude Control and Momentum Management," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 6, 1990, pp. 993–999.

[3]Bennett, W. H., Kwatny, H. G., LaVigna, C., and Blankenship, G. L., "Nonlinear and Adaptive Control of Flexible Space Structures," *Control of Systems With Inexact Dynamic Models,* edited by N. Sadegh and Y. H. Chen, ASME, New York, 1991, pp. 73–81.

[4]Rodriguez, G., "Kalman Filtering, Smoothing and Recursive Robot Arm Forward and Inverse Dynamics," *IEEE Journal of Robotics and Automation,* Vol. RA-3, No. 6, 1987, pp. 624–639.

[5]Jain, A., "Unified Formulation of Dynamics for Serial Rigid Multibody Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 3, 1991, pp. 531–542.

[6]Jain, A., and Rodriguez, G., "Recursive Flexible Multibody System Dynamics Using Spatial Operators," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 6, 1992, pp. 1453–1466.

[7]Arnold, V. I., Kozlov, V. V., and Neishtadt, A. I., *Mathematical Aspects of Classical and Celestial Mechanics, Encyclopedia of Mathematical Sciences,* edited by V. I. Arnold, Vol. 3, Springer-Verlag, Heidelberg, 1988.

[8]Chetaev, N. G., *Theoretical Mechanics,* Springer-Verlag, New York, 1989.

[9]Meirovitch, L., *Methods of Analytical Dynamics,* McGraw-Hill, New York, 1970.

[10]Rosenberg, R. M., *Analytical Dynamics of Discrete Systems,* Plenum Press, New York, 1977.

[11]Neimark, J. I., and Fufaev, N. A., *Dynamics of Nonholonomic Systems,* Translations of Mathematical Monographs, Vol. 33, American Mathematical Society, Providence, RI, 1972.

[12]Kurdila, A., Papastravidis, J. G., and Kamat, M. P., "Role of Maggis Equations in Computational Methods for Constrained Multibody Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 1, 1990, pp. 113–120.

[13]Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications,* McGraw-Hill, New York, 1985.

[14]Chetaev, N. G., "On the Equations of Poincaré," *PMM (Applied Mathematics and Mechanics),* No. 5, 1941, pp. 253–262.

[15]Kwatny, H. G., Massimo, F. M., and Bahar, L. Y., "The Generalized Lagrange Equations for Nonlinear RLC Networks," *IEEE Transactions on Circuits and Systems,* Vol. CAS-29, No. 4, 1982, pp. 220–233.

[16]Isidori, A., *Nonlinear Control Systems,* Springer-Verlag, New York, 1989.

[17]Dwyer, T. A. W., "Exact Nonlinear Control of Large Rotational Maneuvers," *IEEE Transactions on Automatic Control,* Vol. AC-29, No. 9, 1984, pp. 539–542.

[18]Sastry, S. S., and Isidori, A., "Adaptive Control of Linearizable Systems," *IEEE Transactions on Automatic Control,* Vol. 34, No. 11, 1989, pp. 1123–1131.